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SEMI-ANALYTICAL APPROACH TO SOLVE NON-LINEAR DIFFERENTIAL EQUATIONS AND THEIR GRAPHICAL REPRESENTATIONS

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ABSTRACT

In this paper we have applied a new approximation technique to solve non-linear partial differential equation. The approximation technique is called Homotopy perturbation method (HPM). The main difference between traditional perturbation method and this one is that it can be applied for even higher values of the parameter, where as traditional perturbation method can be applied only for lower values of parameter. It means that when the value of the parameter is less than one only then the traditional perturbation can be applied. In this paper we have considered a highly non-linear partial differential equation and found the approximate solution using HPM for two types of initial conditions. Then we have drawn two-dimensional and three dimensional graphs from the solution of the equations which demonstrates the physical situation of the solution for different values of the parameter. This gives us the clear picture of the range of the variables for which the normal solution exists and for what values of the variable the chaotic situation arises.

KEYWORDS: Approximation Solution, Chaotic Solution, Homotopy, Homotopy Perturbation, Perturbation

1. INTRODUCTION

Chaos or chaotic system received great attention among mathematicians and physicists. Because it stemps out from natural phenomena. Mathematically one can get this studying linear or non-linear dynamical system or non-linear partial differential equations. Unfortunately to get a close form of the solution of any of them is almost beyond our present knowledge. However, one can use some kind of analytical or numerical methods to get an approximate solution.

Recently, J. H. He [1, 2] found an ingenious method to solve non-linear differential equation, which is called "Homotopy Perturbation Method" (HPM). He applied this method to solve different types of non-linear equations, such as Van der Pol- duffing equations [3] for different types of oscillator [4, 5, 6] problems and so on.

Here we have applied HPM technique to find the analytical solution of a non-linear differential equation with different types of initial conditions. Taking different values of the parameters and for a wide range of time we have found the graphical representation of the analytical solution for all domain of response. Following He's paper [2] we have presented the basic prescription of the method in section 2. In section 3 we have applied this technique to solve non-linear partial differential equation with two different types of initial conditions and found an approximate solution of the problem. In section 4 we have given the graphical representation of the solution for two different initial conditions. In last section we have discussed the nature of the solution for different values of the parameters and range of variables

Homotopy Perturbation Prescription

J. H. He [1] gave a prescription to solve nonlinear differential equation which can be expressed in the following way.

Let us consider a nonlinear differential equation of the form

$$A(u) - f(r) = 0,\ r \in \Omega \tag{1}$$

Subject to the boundary conditions:

$$B(u, \partial u/\partial n) = 0,\ r \in \Gamma \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω .

In general one can divide the operator A into two parts: linear and non- linear. That means

$$A = L + N$$

where L is linear and N is non-linear.

Hence, equation (1) can now be rewritten as

$$L(u) + N(u) - f(r) = 0,\ r \in \Omega \tag{3}$$

By the homotopy technique, one can construct a homotopy in the following way $v(r, p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0,\ p \in [0,1], r \in \Omega \tag{4}$$

or $H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0$ (5)

Where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (1) which satisfies the boundary conditions.

Obviously, from equations (4) and (5) we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{7}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of equations (4) and (5) can be written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + \dots\dots\dots \tag{8}$$

Setting $p = 1$ results in the approximate solution of equation (1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots\dots\dots \tag{9}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques. The series (9) is convergent for most cases.

2. SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATION

In this section we will show how He's homotopy method can be applied non-linear partial differential equation with different types of initial conditions. In this view we have considered a highly non-linear partial differential equation with two types of initial conditions, and used He's homotopy method to solve these problems. The two different types of initial conditions considered here is just to see the sensitivity of the behaviour of the solutions for different types of initial conditions. Because in any dynamical system it is very important to check whether this types of equations is sensitive to initial conditions.

3.1. Solution of Non-Linear Equation with Initial Conditions

Let us consider a highly non-linear partial differential equation

$$U_{tt} + \alpha U_{xx} + \beta U + \gamma U^3 = 0 \tag{10}$$

and try to find the solution with the following initial conditions:

$$U(x,0) = x^3 \text{ and } U_t(x,0) = 0 \tag{11}$$

In order to solve equation (10) using HPM, a homotopy- perturbation method can be constructed as follows

$$H(v,p) = (1-p)\frac{\partial^2 v}{\partial t^2} + p(\frac{\partial^2 v}{\partial t^2} + \alpha \frac{\partial^2 v}{\partial x^2} + \beta v + \gamma v^3) = 0 \tag{12}$$

Substituting $u = \lim_{p\to 1} v = \lim_{p\to 1}[v_0 + p v_1 + p^2 v_2 + \dots\dots\dots$ into equation (12) and rearranging the resultant equation based on powers of p -terms, we can find the following equations:

$$p^0 : \frac{\partial^2 v_0}{\partial t^2} = 0 \tag{13}$$

$$p^1 : \frac{\partial^2 v_1}{\partial t^2} + \alpha \frac{\partial^2 v_0}{\partial x^2} + \beta v_0 + \gamma {v_0}^3 = 0 \tag{14}$$

$$p^2 : \frac{\partial^2 v_2}{\partial t^2} + \alpha \frac{\partial^2 v_1}{\partial x^2} + \beta v_1 + \gamma {v_0}^2 v_1 = 0 \tag{15}$$

Solving equation (13) we get

$$\frac{\partial v_0}{\partial t} = A \tag{16}$$

$$v_0 = At + B \tag{17}$$

Before using the initial condition in (16) and (17) let us clarify one important point. If we back to earlier substitution that is

$$u(x,t) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots\dots\dots$$

and taking the initial conditions : $U(x,0) = x^3$ and $U_t(x,0) = 0$

which implies, $v_0 = x^3$ and $v_1 = v_2 = \dots\dots\dots = 0$

Using the initial conditions (11), at $t = 0$ we get $A = 0$ and $B = x^3$

Hence, finally we get the solution

$$v_0 = x^3 \tag{18}$$

From this equation we find

$$\frac{\partial v_0}{\partial x} = 3x^2,\ \frac{\partial^2 v_0}{\partial t^2} = 6x$$

After substituting the value of v_0, $\frac{\partial v_0}{\partial x}$ and $\frac{\partial^2 v_0}{\partial t^2}$ in (14)

we get the equation for v_1

$$\frac{\partial^2 v_1}{\partial t^2} = -(6\alpha x + \beta x^3 + \gamma x^9)$$

Integrating

$$\frac{\partial v_1}{\partial t} = -(6\alpha x + \beta x^3 + \gamma x^9)t + C$$

Again integrating we arrived at

$$v_1 = -\frac{1}{2}(6\alpha x + \beta x^3 + \gamma x^9)t^2 + Ct + D$$

Using the initial conditions in the above equations we get

at $t = 0$, $\frac{\partial v_1}{\partial t} = 0 = C$, $v_1(x,0) = 0 = D$

Putting the values of C and D we get

$$v_1 = -\frac{1}{2}(6\alpha x + \beta x^3 + \gamma x^9)t^2 \tag{19}$$

which is the solution for v_1

Now to solve equation (15) for v_2 let us find

$$\frac{\partial v_1}{\partial x} = -\frac{1}{2}(6\alpha + 3\beta x^2 + 9\gamma x^8)t^2$$

$$\frac{\partial^2 v_1}{\partial x^2} = -\frac{1}{2}(6\beta x + 72\gamma x^7)t^2$$

Using these in equation (15) we get

$$\frac{\partial^2 v_2}{\partial t^2} = -(\alpha \frac{\partial^2 v_1}{\partial x^2} + \beta v_1 + 3\gamma v_0{}^2 v_1)$$

$$= \frac{1}{2}(12\alpha\beta x + \beta^2 x^3 + 90\alpha\gamma x^7 + 4\beta\gamma x^9 + 3\gamma^2 x^{15})t^2$$

Integrating,

$$\frac{\partial v_2}{\partial t} = \frac{1}{6}(12\alpha\beta x + \beta^2 x^3 + 90\alpha\gamma x^7 + 4\beta\gamma x^9 + 3\gamma^2 x^{15})t^3 + C'$$

Again integrating,

$$v_2 = \frac{1}{24}(12\alpha\beta x + \beta^2 x^3 + 90\alpha\gamma x^7 + 4\beta\gamma x^9 + 3\gamma^2 x^{15})t^4 + C't + D'$$

Again the initial conditions (11) implies that

$C' = 0$ and $D' = 0$

Therefore

$$v_2 = \frac{1}{24}(12\alpha\beta x + \beta^2 x^3 + 90\alpha\gamma x^7 + 4\beta\gamma x^9 + 3\gamma^2 x^{15})t^4 \tag{20}$$

According to HPM we can write the solution of (10) up to second order of p as :

$$u(x,t) = \lim_{p\to 1} v = \lim_{p\to 1}[v_0 + p v_1 + p^2 v_2]$$

Setting $p = 1$ the above equation becomes,

$$u(x,t) = v_0 + v_1 + v_2$$

Substituting the values of v_0, v_1 and v_2

$$u(x,t) = \frac{1}{24}[24x^3 - (72\alpha x + 12\beta x^3 + 12\gamma x^9)t^2$$

$$+(12\alpha\beta x+\beta^2 x^3+90\alpha\gamma x^7+4\beta\gamma x^9+3\gamma^2 x^{15})t^4] \tag{21}$$

This is the approximate solution of the non-linear partial differential equation (10) with initial conditions (11).

3.2. Solution of Same Non-Linear Differential Equation with another Initial Conditions

Again considering the same differential equation

$$U_{tt}+\alpha U_{xx}+\beta U+\gamma U^3=0 \tag{22}$$

with different type of initial conditions:

$$u(x,0)=0 \text{ and } U_t(x,0)=e^{\lambda x} \tag{23}$$

Following the same procedure for this problem we get,

$$U(x,t)=\frac{1}{120}e^{\lambda x}[120t-(20\alpha\lambda^2+20\beta)t^3+(\alpha^2\lambda^4+2\alpha\beta\lambda^2+\beta^2)t^5]$$

$$-\frac{1}{840}e^{3\lambda x}[42\gamma t^5-(19\alpha\gamma\lambda^2+11\beta\gamma)t^7]+\frac{1}{480}\gamma^2 e^{5\lambda x}t^9 \tag{24}$$

This is the approximate analytical solution of the partial differential equation (22) with another type of initial conditions.

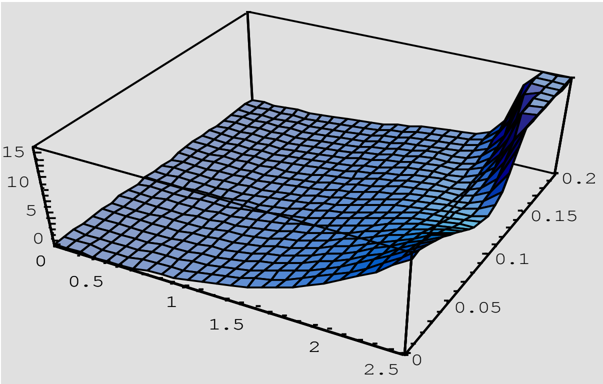
With the help of HPM we are able to solve highly nonlinear higher order partial differential equation with different types of initial conditions.

In the following sections we will find the graphical representation of the solutions of non-linear differential equations (10) and (22) found in (21) and (24) for different values of parameters α, β, γ and for a wide range of x and t values. From which we can draw some conclusion of the results.

4. GRAPHICAL REPRESENTATION OF THE SOLUTIONS OF (10) AND (22)

In this section we have drawn three-dimensional and two-dimensional graphs from the solution (21) of non-linear differential equation (10) with the given initial conditions taking larger and smaller values of the parameter and for a wide range of the values of the variables x and t . After a close look to the graphs we have divided the shape of the graphs in four kinds. The graphs of similar kind for different parametric values but for a particular range of x and t are depicted below.

First Kind



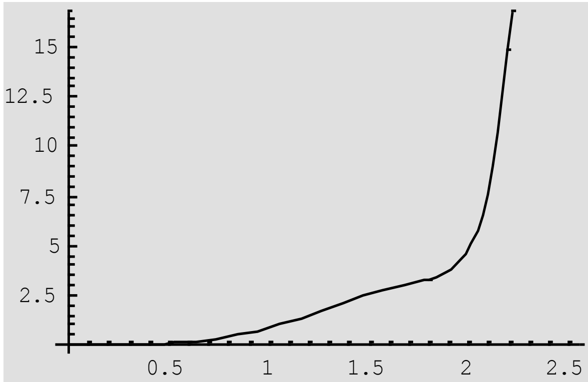
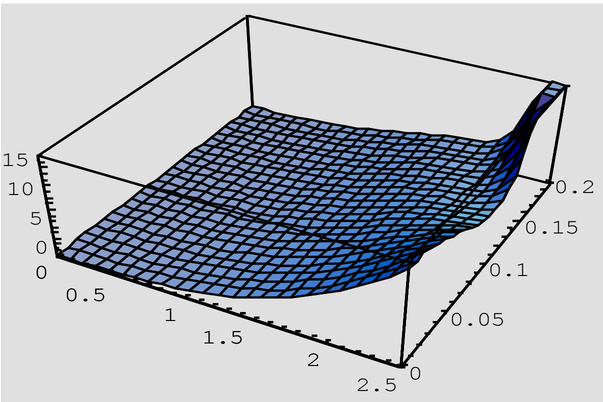


Figure 1 **Figure 1'**

Figure 1: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$ and $\gamma = 1$ and Values of the Variables $x \in (0,2.5)$ and $t \in (0,.2)$. Figure 1' Shows the Corresponding Two-Dimensional Graph



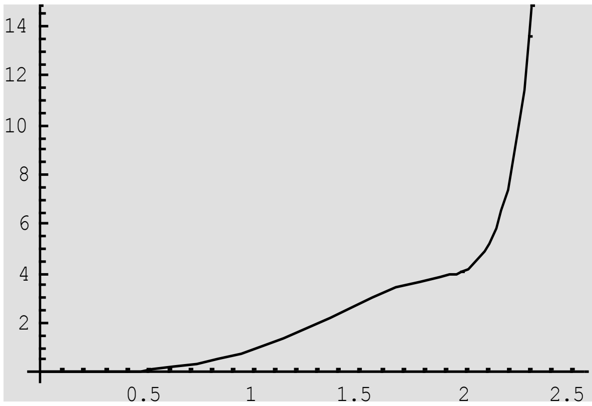
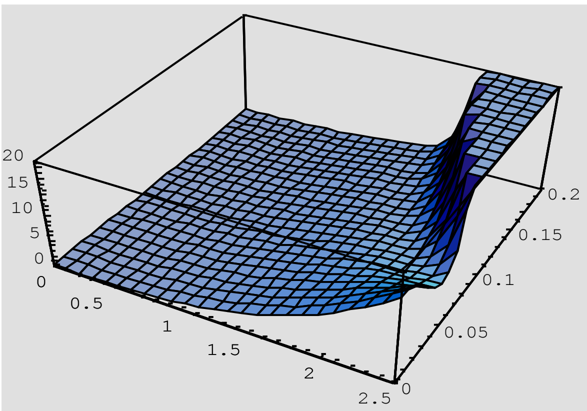


Figure 2 **Figure 2'**

Figure 2: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5$, $\beta = .6$ and $\gamma = .7$ and Values of the Variables $x \in (0,2.5)$ and $t \in (0,.2)$. Figure 2' Shows the Corresponding Two-Dimensional Graph



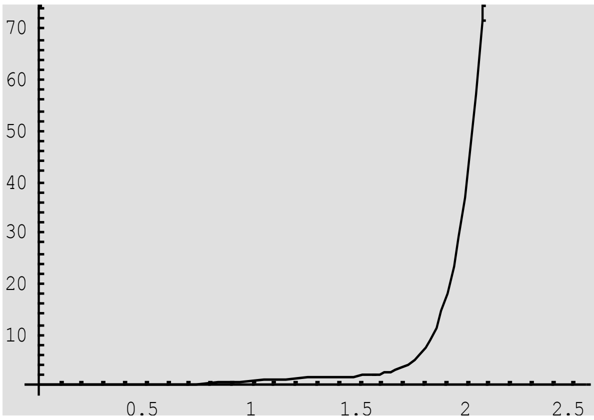
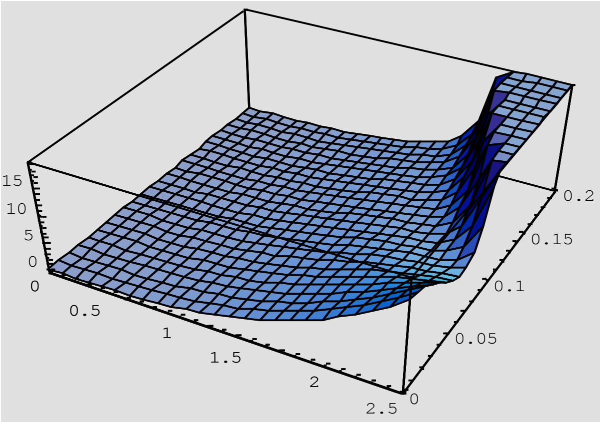


Figure 3 **Figure 3'**

Figure 3: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 2$ and $\gamma = 3$ and Values of the Variables $x \in (0,2.5)$ and $t \in (0,.2)$. Figure 3' Shows the Corresponding Two-Dimensional Graph



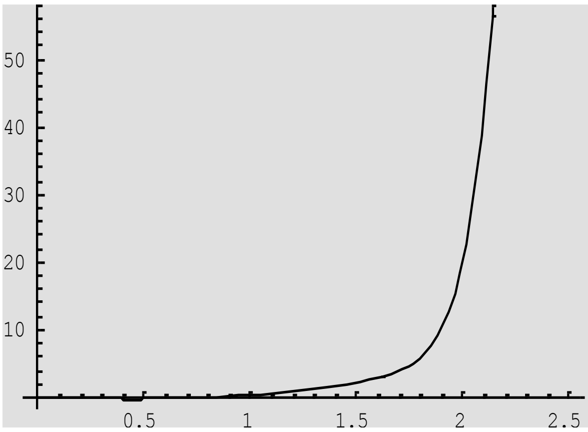
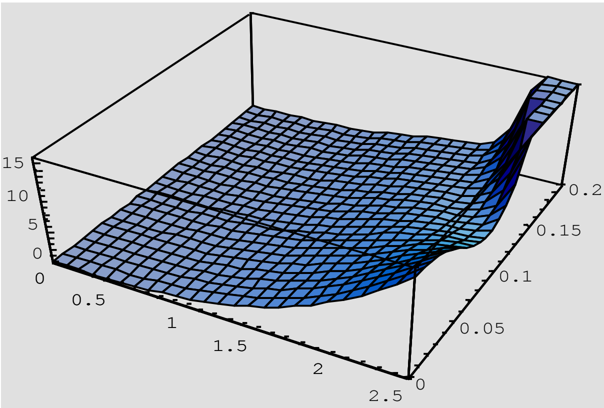


Figure 4 **Figure 4'**

Figure 4: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 5, \beta = 1$ and $\gamma = 2$ and Values of the Variables $x \in (0,2.5)$ and $t \in (0,.2)$. Figure 4' Shows the Corresponding Two-Dimensional Graph



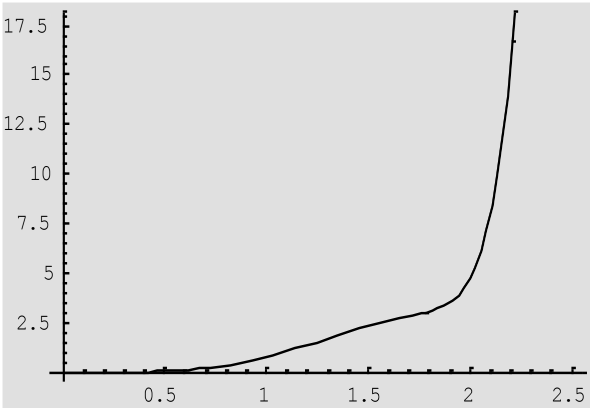
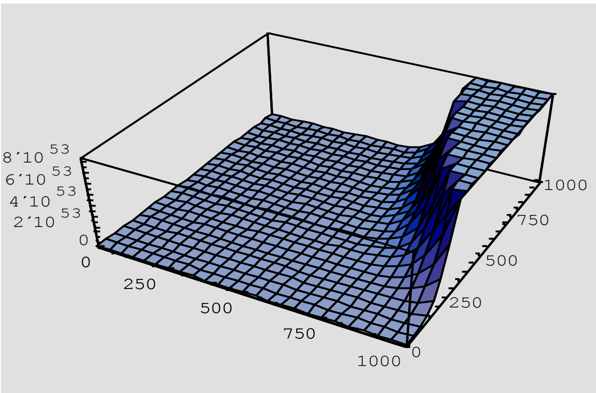


Figure 5 **Figure 5'**

Figure 5: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 5$ and $\gamma = 1$ and Values of the Variables $x \in (0,2.5)$ and $t \in (0,.2)$. Figure 5' Shows the Corresponding Two-Dimensional Graph

Second Kind



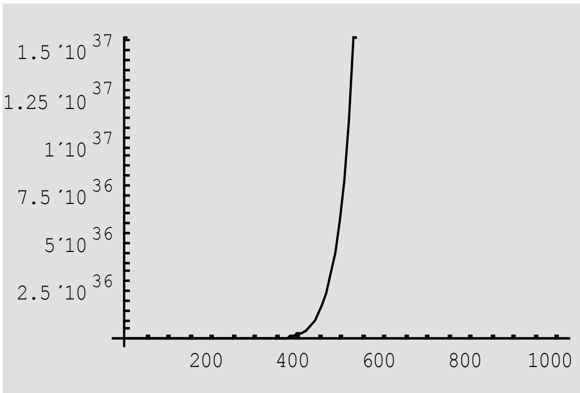
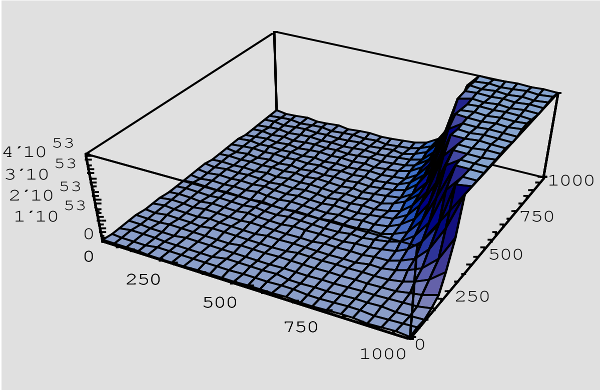


Figure 6 **Figure 6'**

Figure 6: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 1$ and $\gamma = 1$ and Values of the Variables $x \in (0,1000)$ and $t \in (0,1000)$ Figure 6' Shows the Corresponding Two-Dimensional Graph



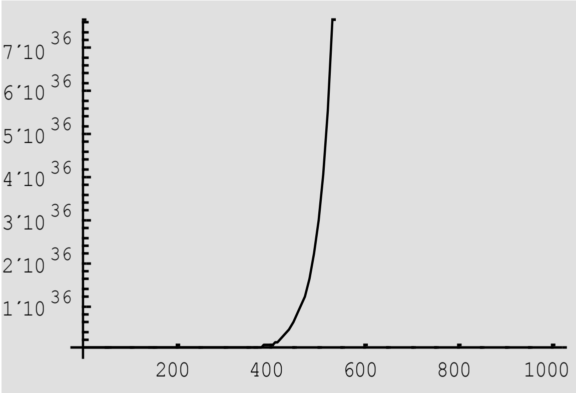
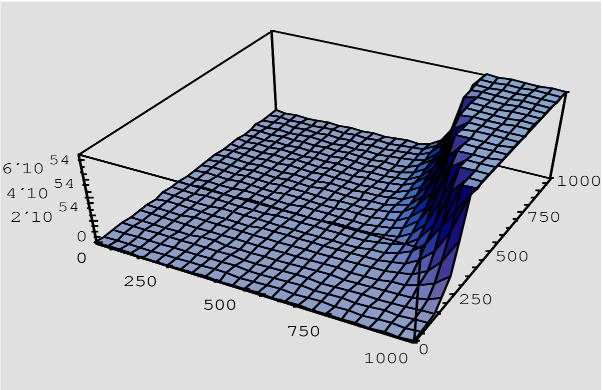


Figure 7 **Figure 7'**

Figure 7: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5$, $\beta = .6$ and $\gamma = .7$ and Values of the Variables $x \in (0,1000)$and $t \in (0,1000)$. Figure 7' Shows the Corresponding Two-Dimensional Graph



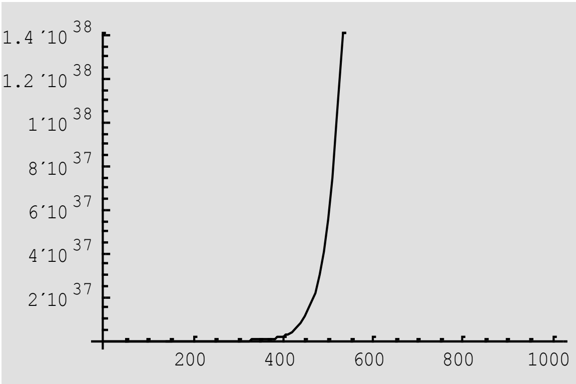
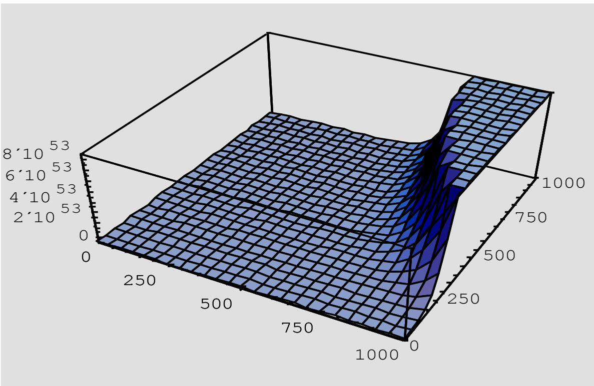


Figure 8 **Figure 8'**

Figure 8: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 2$ and $\gamma = 3$ and Values of the Variables $x \in (0,1000)$and $t \in (0,1000)$. Figure 8' Shows the Corresponding Two-Dimensional Graph



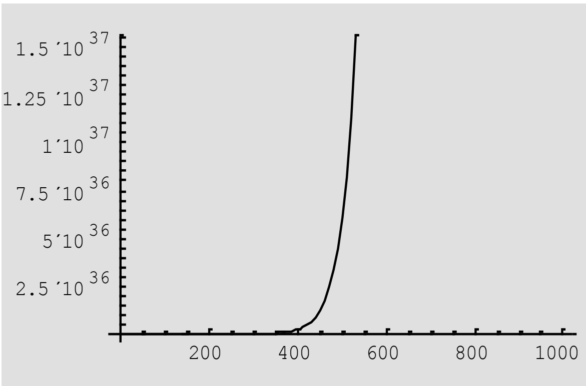
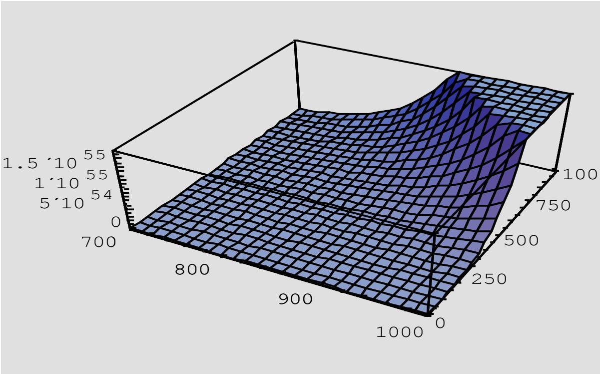


Figure 9 **Figure 9'**

Figure 9: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 5$ and $\gamma = 1$ and Values of the Variables $x \in (0,1000)$and $t \in (0,1000)$. Figure 9' Shows the Corresponding Two-Dimensional Graph

Third Kind



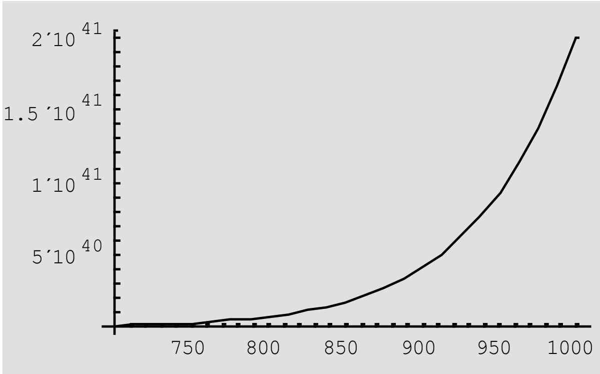
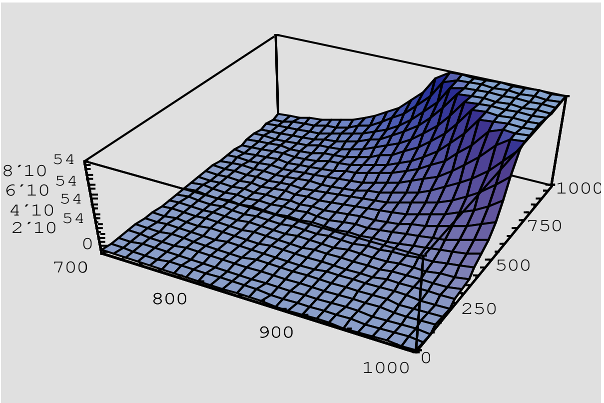


Figure 10 **Figure 10'**

Figure 10: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$ **and** $\gamma = 1$ **and Values of the Variables** $x \in (700, 1000)$ **and** $t \in (0, 1000)$**. Figure 10' Shows the Corresponding Two-Dimensional Graph**



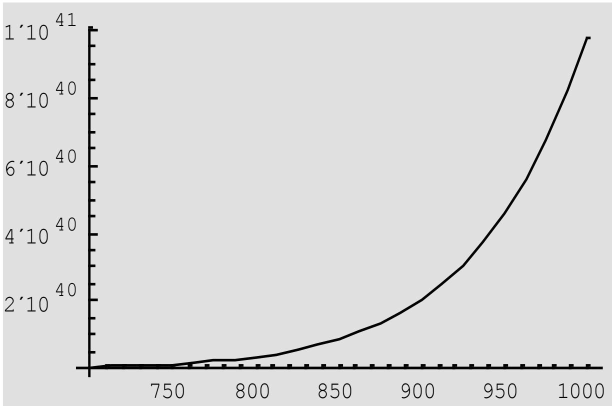
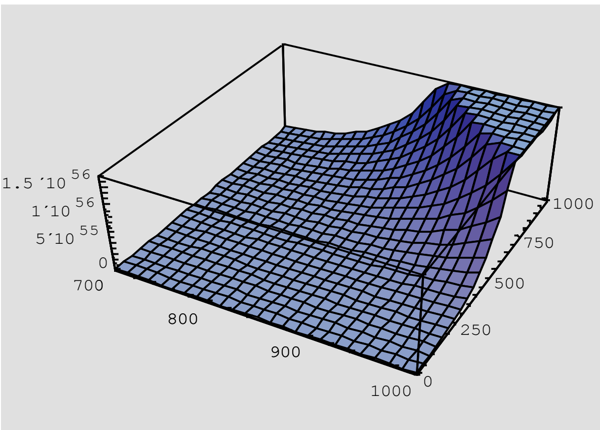


Figure 11 **Figure 11'**

Figure 11: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5$, $\beta = .6$ **and** $\gamma = .7$ **and Values of the Variables** $x \in (700, 1000)$**and** $t \in (0, 1000)$**. Figure 11' Shows the Corresponding Two-Dimensional Graph**



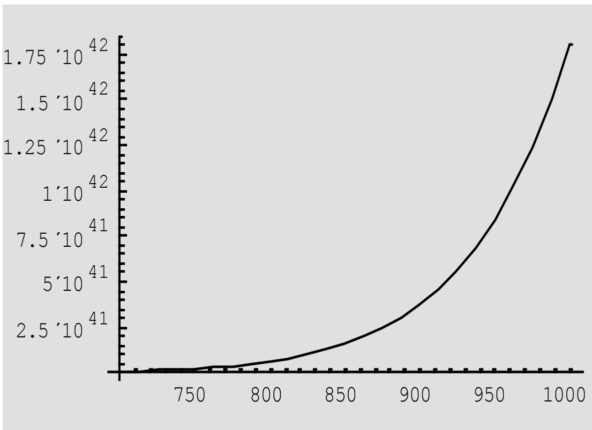
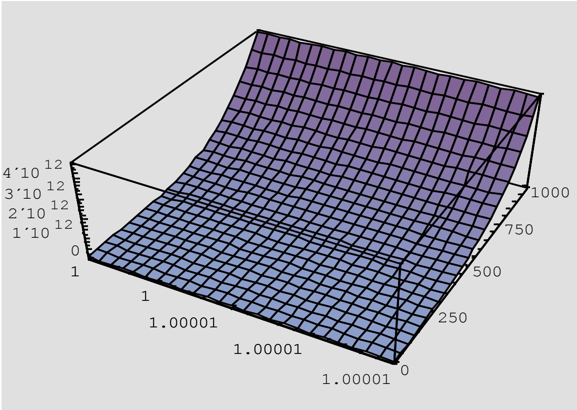


Figure 12 **Figure 12'**

Figure 12: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 2$ **and** $\gamma = 3$ **and Values of the Variables** $x \in (700, 1000)$ **and** $t \in (0, 1000)$**. Figure 12' Shows the Corresponding Two-Dimensional Graph**

Fourth Kind



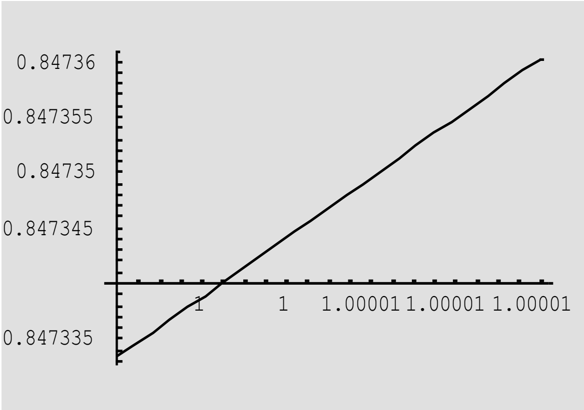
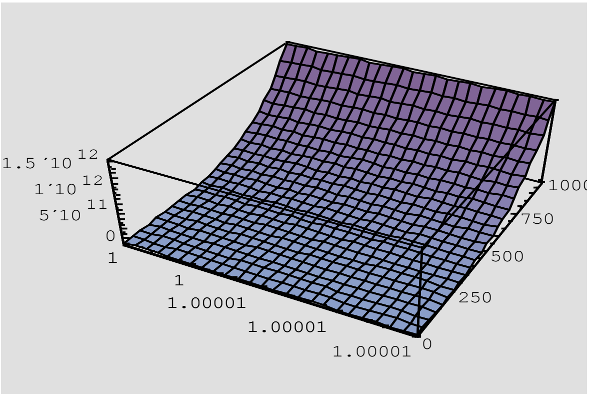


Figure 13 **Figure 13'**

Figure 13: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$ and $\gamma = 1$ and Values of the variables $x \in (1, 1.00001)$and $t \in (0, 1000)$. Figure 13' Shows the Corresponding Two-Dimensional Graph



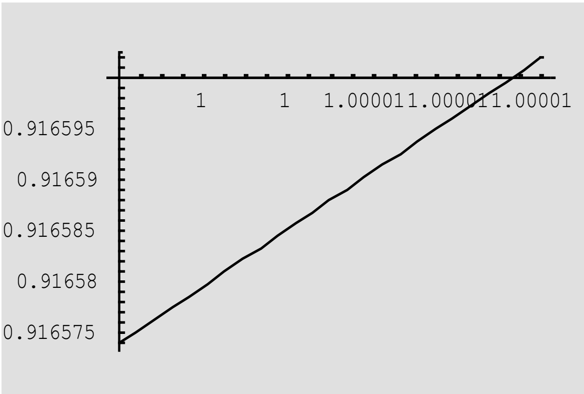
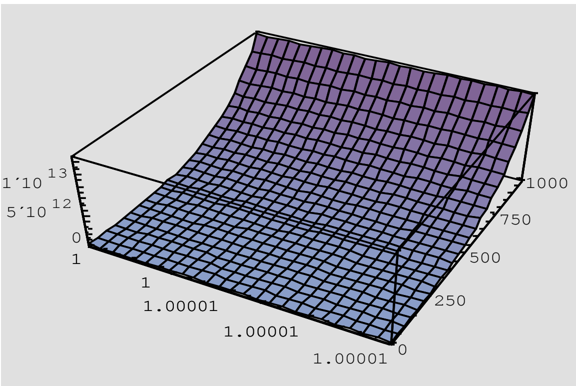


Figure 14 **Figure 14'**

Figure 14: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5$, $\beta = .6$ and $\gamma = .7$ and Values of the Variables $x \in (1, 1.00001)$and $t \in (0, 1000)$. Figure 14' shows the Corresponding Two-Dimensional Graph



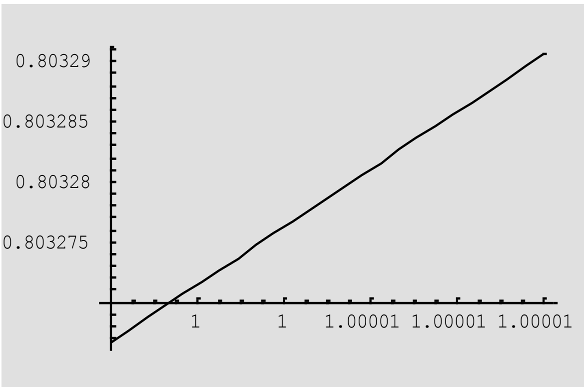
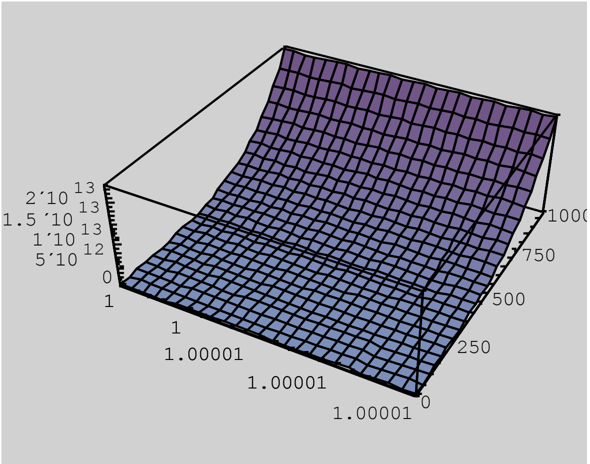


Figure 15 **Figure 15'**

Figure 15: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 2$ and $\gamma = 3$ and Values of the Variables $x \in (1, 1.00001)$and $t \in (0, 1000)$. Figure 15' Shows the Corresponding Two-Dimensional Graph



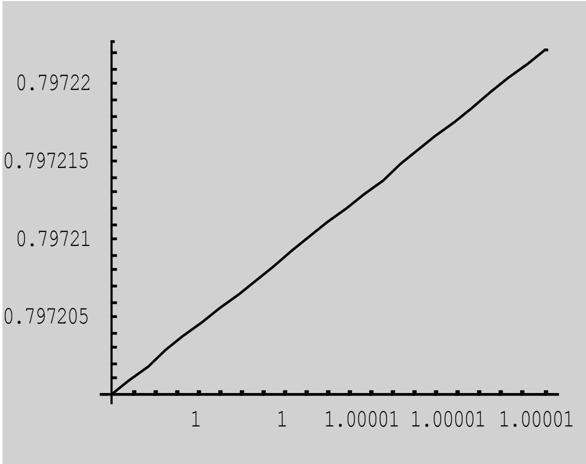


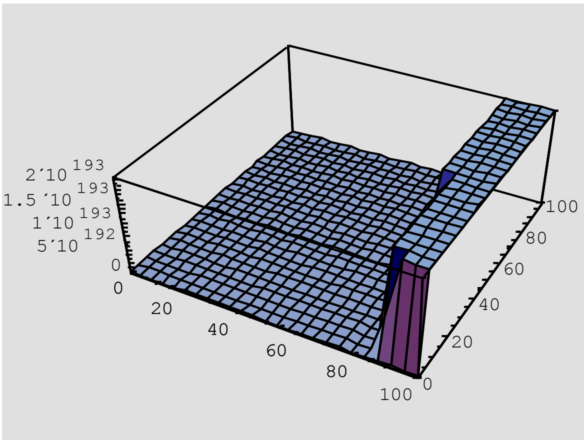
Figure 16 **Figure 16'**

Figure 16: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = -1$, $\beta = -1$ and $\gamma = -5$ and Values of the Variables $x \in (1, 1.00001)$and $t \in (0, 1000)$. Figure 16' Shows the Corresponding Two-Dimensional Graph

Analysis of the Solution of Equation (22)

The solution of equation (22) is given in equation (24). From this solution we have drawn many three-dimensional and two dimensional graphs for different values of the parameters and for a wide range of the variables. The significant graphs are shown below. If we look very carefully these graphs, we can see that the shape of the graphs changes for different range of variables. A slight change of the graphs has also been seen for positive and negative values of the parameters.

First Kind



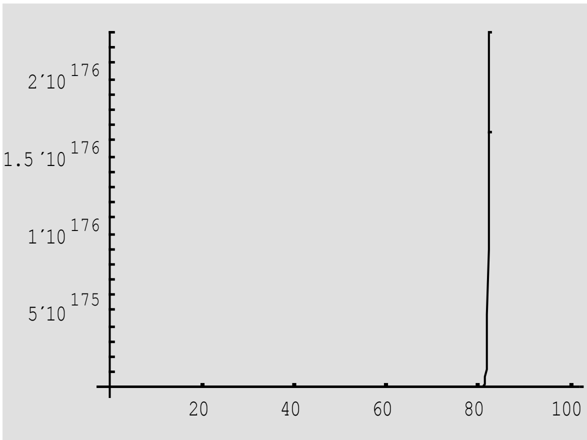
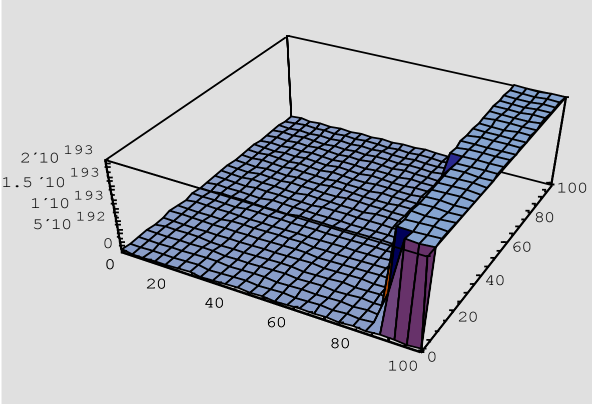


Figure 17 **Figure 17'**

Figure 17: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$, $\gamma = 1$and $\lambda = 1$ and Values of the Variables $x \in (0, 100)$and $t \in (0, 100)$. Figure 17' Shows the Corresponding Two-Dimensional Graph



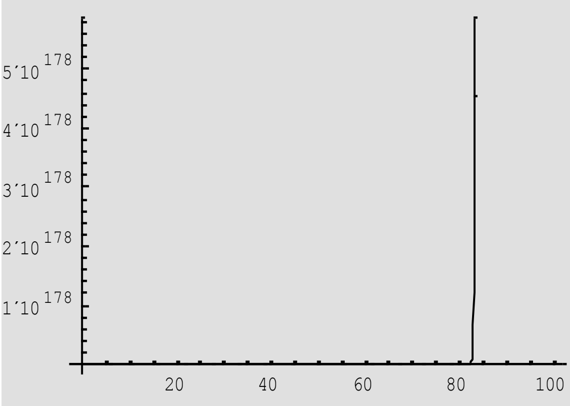
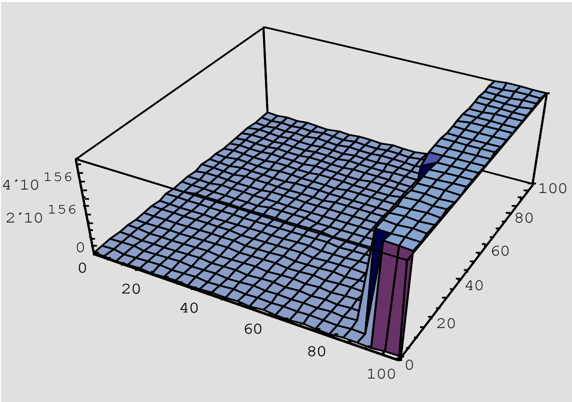


Figure 18 **Figure 18'**

Figure 18: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha=-1, \beta=1$, $\gamma=1$ and $\lambda=1$ and Values of the Variables $x\in(0,100)$ and $t\in(0,100)$. Figure 18' Shows the Corresponding Two-Dimensional Graph



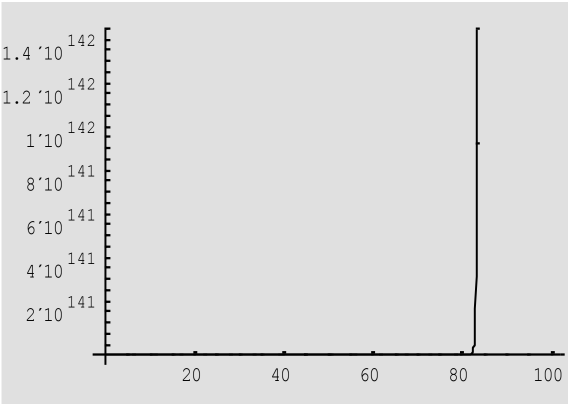
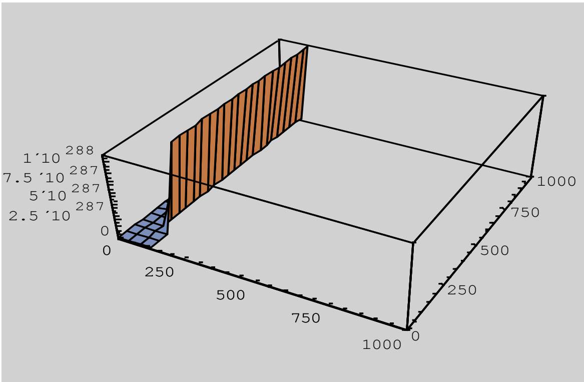


Figure 19 **Figure 19'**

Figure 19: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha=.5$, $\beta=.6$, $\gamma=.7$ and $\lambda=.8$ and Values of the Variables $x\in(0,100)$ and $t\in(0,100)$. Figure 5.24' Shows the Corresponding Two-Dimensional Graph

Second Kind



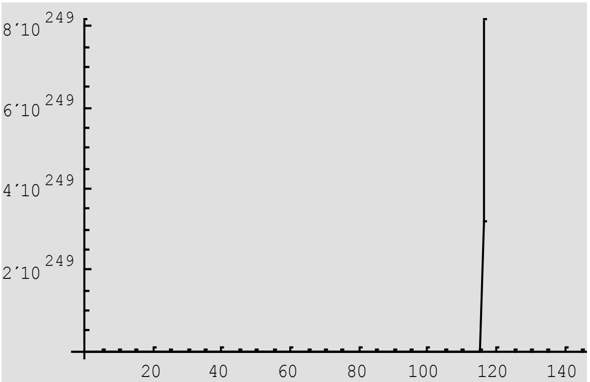
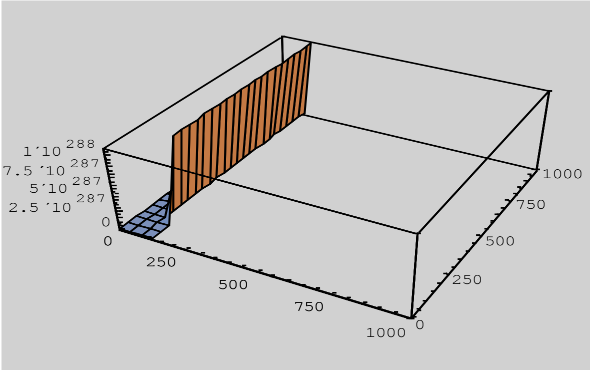


Figure 20 **Figure 20'**

Figure 20: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha=1$, $\beta=1$, $\gamma=1$ and $\lambda=1$ and Values of the Variables $x\in(0,1000)$ and $t\in(0,1000)$. Figure 20' Shows the Corresponding Two-Dimensional Graph



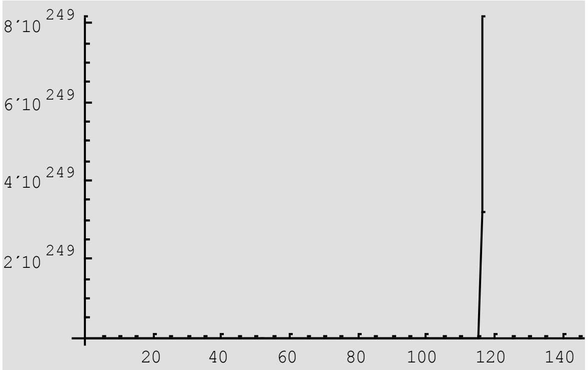
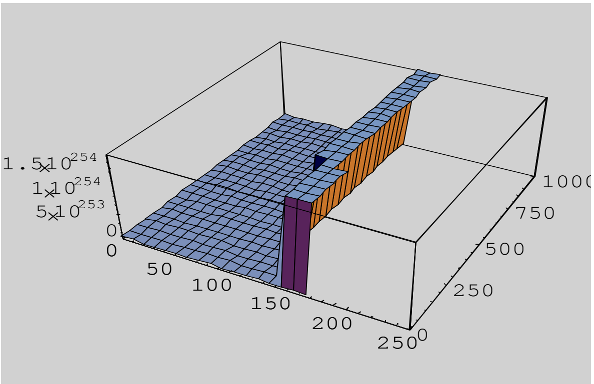


Figure 21 **Figure 21'**

Figure 21: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = -1$**,** $\beta = 1$ **,** $\gamma = 1$ **and** $\lambda = 1$ **and Values of the Variables** $x \in (0,1000)$ **and** $t \in (0,1000)$**. Figure 21' Shows the Corresponding Two-Dimensional Graph**

Third Kind



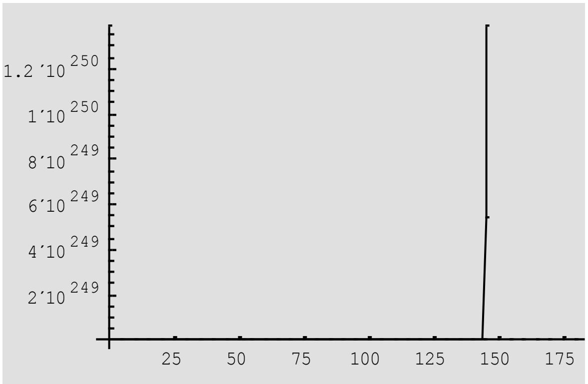
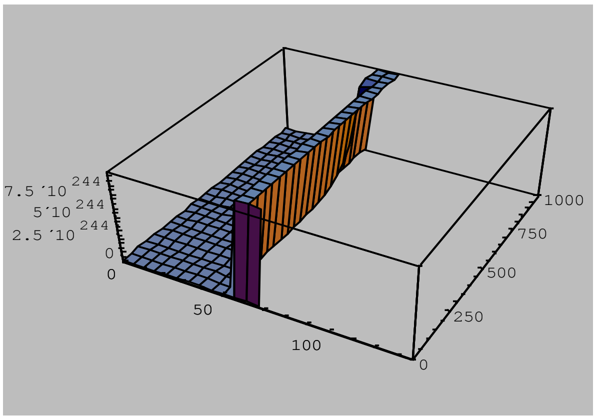


Figure 22 **Figure 22'**

Figure 22: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5, \beta = .6$ **,** $\gamma = .7$ **and** $\lambda = .8$ **and Values of the Variables** $x \in (0,250)$ **and** $t \in (0,1000)$**. Figure 22' Shows the Corresponding Two-Dimensional Graph**

Fourth Kind



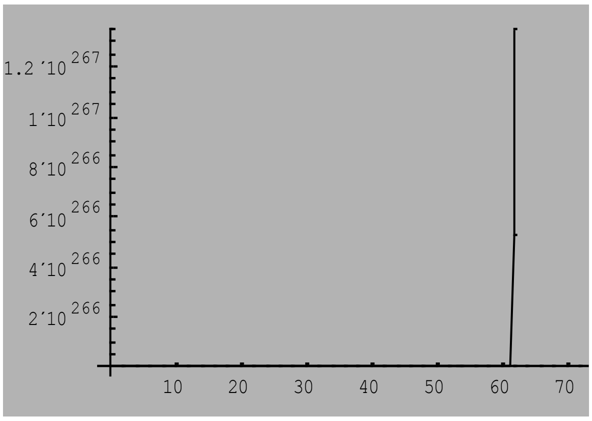
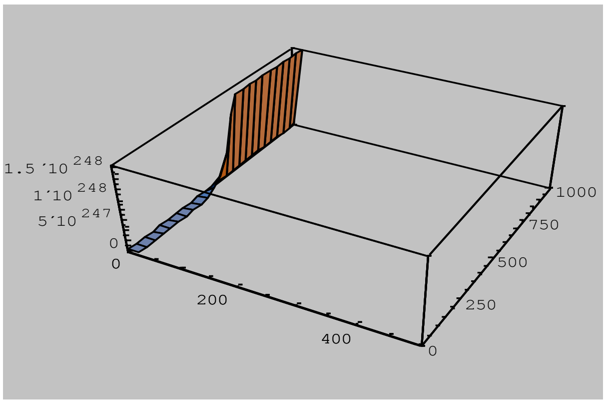


Figure 23 **Figure 23'**

Figure 23: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5, \beta = .6$ **,** $\gamma = .7$ **and** $\lambda = 2$ **and Values of the Variables** $x \in (0,136)$ **and** $t \in (0,1000)$**. Figure 23' Shows the Corresponding Two-Dimensional Graph**



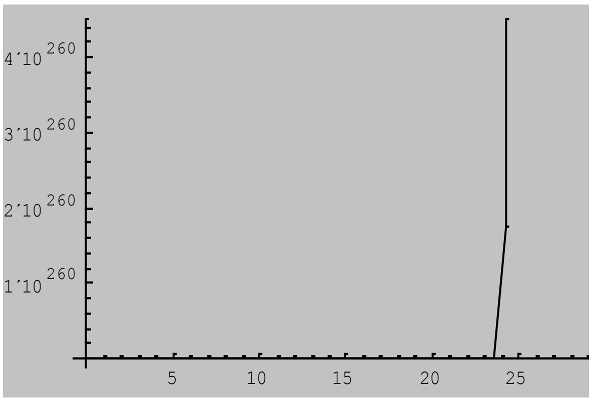
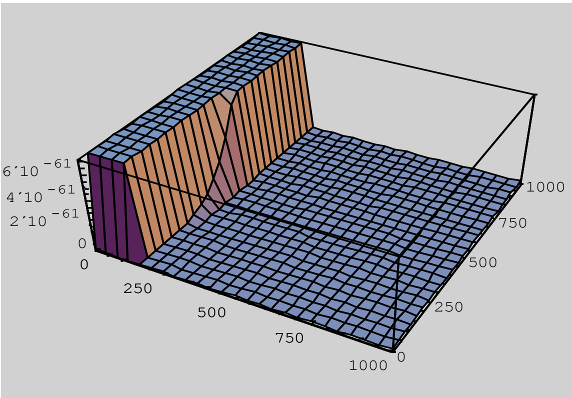


Figure 24 **Figure 24'**

Figure 24: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 1$, $\gamma = 1$ and $\lambda = 5$ and Values of the Variables $x \in (0,500)$ and $t \in (0,1000)$. Figure 24' Shows the Corresponding Two-Dimensional Graph

Fifth Kind



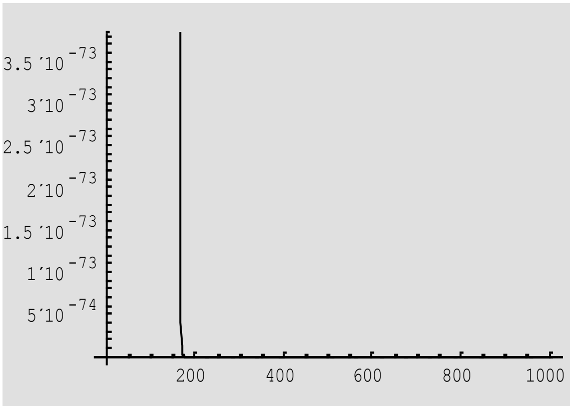
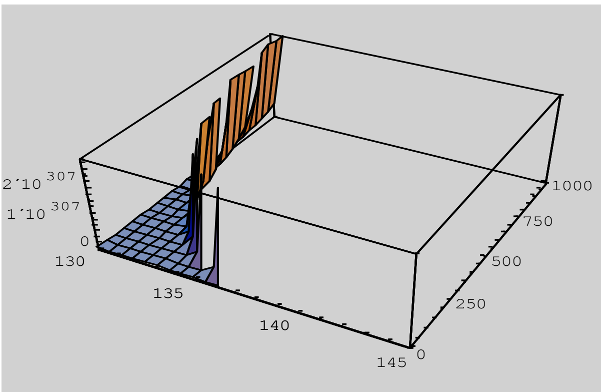


Figure 25 **Figure 25'**

Figure 25: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 1$, $\gamma = 1$ and $\lambda = -1$ and Values of the Variables $x \in (0,1000)$ and $t \in (0,1000)$. Figure 25' Shows the Corresponding Two-Dimensional Graph

Sixth Kind



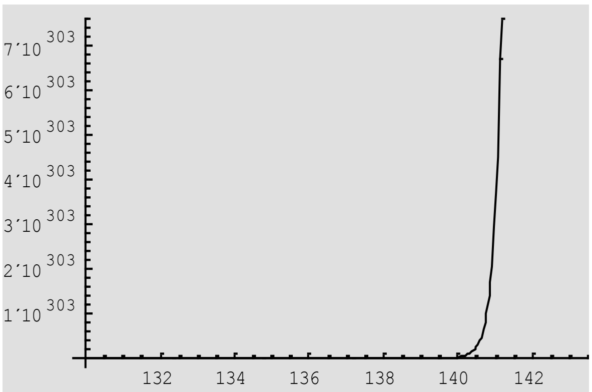
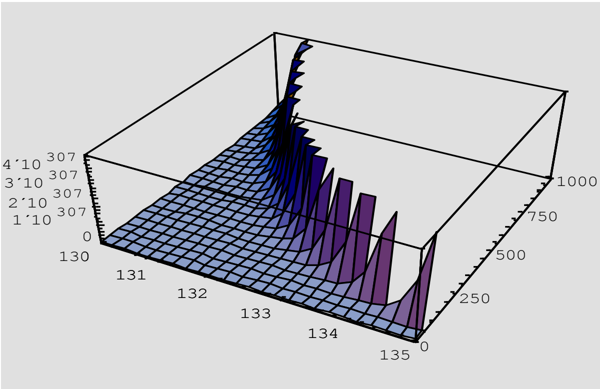


Figure 26 **Figure 26'**

Figure 26: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and $\lambda = 1$ and Values of the Variables $x \in (130,145)$ and $t \in (0,1000)$. Figure 26' Shows the Corresponding Two-Dimensional Graph

Seventh Kind



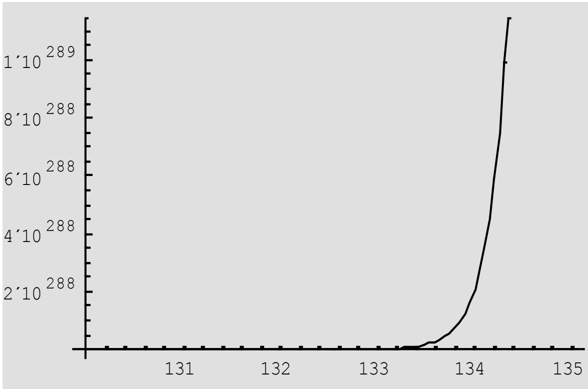
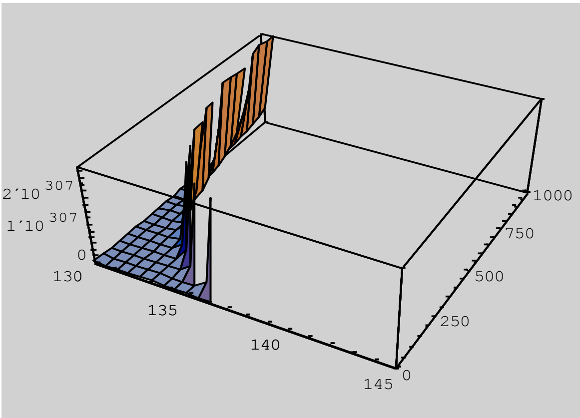


Figure 27 **Figure 27'**

Figure 27: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 1$ **,** $\gamma = 1$ **and** $\lambda = 1$ **and Values of the Variables** $x \in (130, 135)$**and** $t \in (0, 1000)$**. Figure 27' Shows the Corresponding Two-Dimensional Graph**



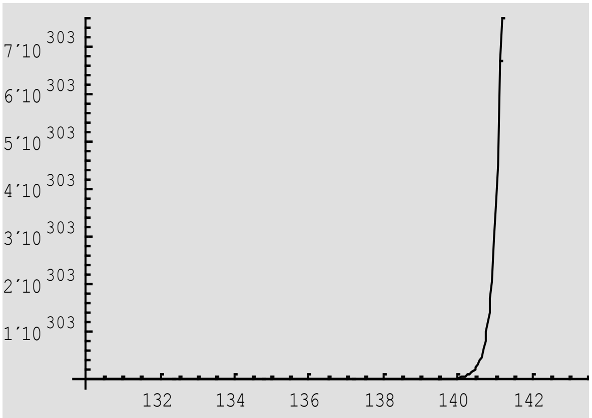
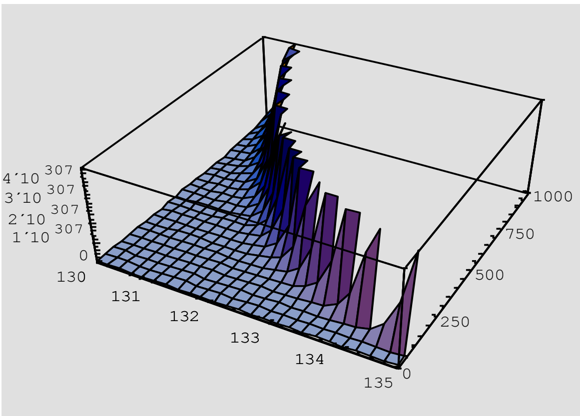


Figure 28 **Figure 28'**

Figure 28: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = -1, \beta = 1$ **,** $\gamma = 1$ **and** $\lambda = 1$ **and Values of the Variables** $x \in (130, 145)$**and** $t \in (0, 1000)$**. Figure 28' Shows the Corresponding Two-Dimensional Graph**



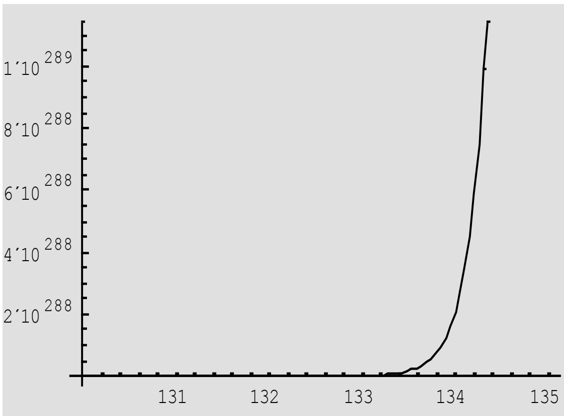
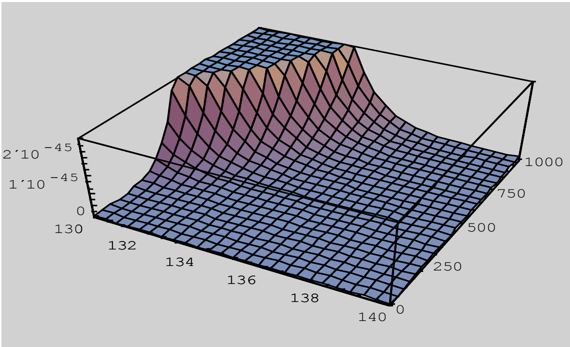


Figure 29 **Figure 29'**

Figure 29: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = -1$ **,** $\beta = 1$ **,** $\gamma = 1$ **and** $\lambda = 1$ **and Values of the Variables** $x \in (130, 135)$**and** $t \in (0, 1000)$**. Figure 29' Shows the Corresponding Two-Dimensional Graph**

Eighth Kind



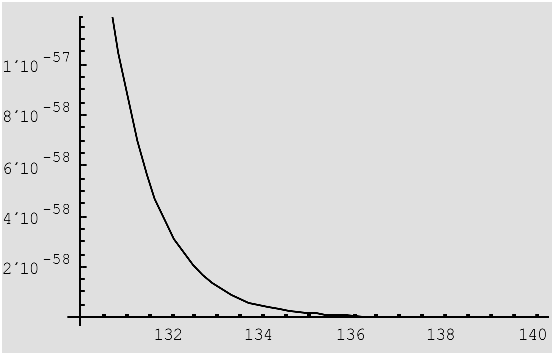
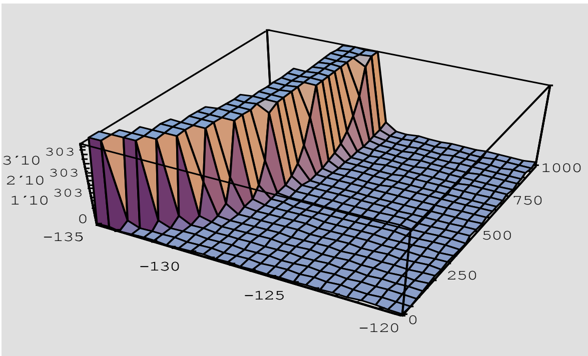


Figure 30 **Figure 30'**

Figure 30: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and $\lambda = -1$ and Values of the Variables $x \in (130,140)$ and $t \in (0,1000)$. Figure 30' Shows the Corresponding Two-Dimensional Graph

Ninth Kind



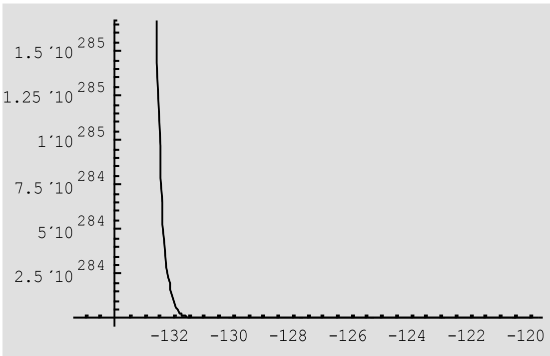
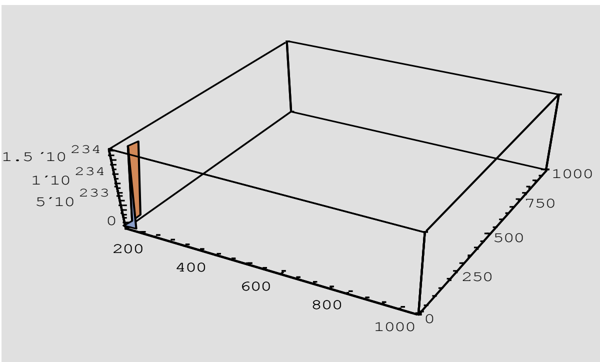


Figure 31 **Figure 31'**

Figure 31: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 1$, $\gamma = 1$ and $\lambda = -1$ and Values of the Variables $x \in (-135,-120)$ and $t \in (0,1000)$. Figure 31' Shows the Corresponding Two-Dimensional Graph

Tenth Kind



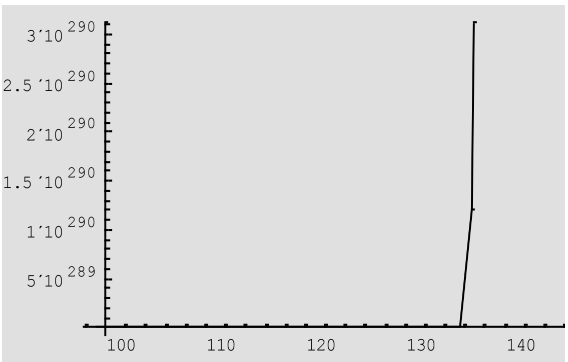
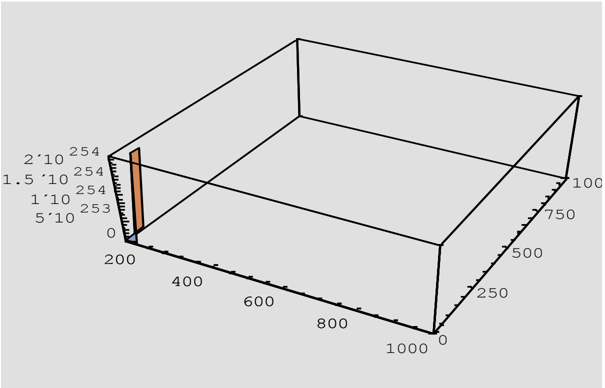


Figure 32 **Figure 32'**

Figure 32: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and $\lambda = 1$ and Values of the Variables $x \in (97,1000)$ and $t \in (0,1000)$. Figure 32' Shows the Corresponding Two-Dimensional Graph



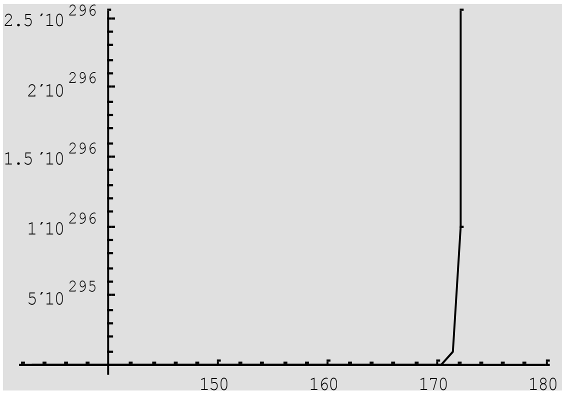
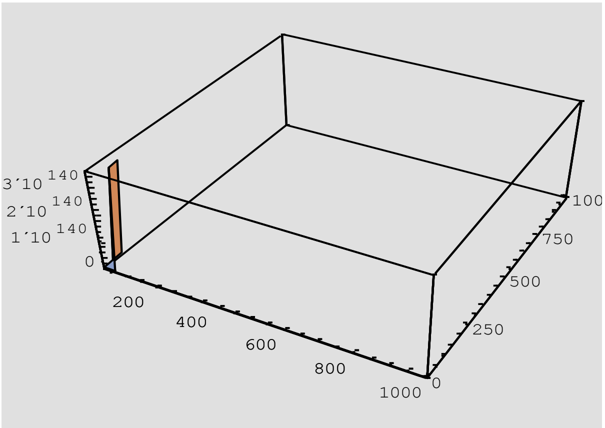


Figure 33 **Figure 33'**

Figure 33: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5$, $\beta = .6$, $\gamma = .7$ and $\lambda = .8$ and Values of the Variables $x \in (133,1000)$and $t \in (0,1000)$. Figure 33'Shows the Corresponding Two-Dimensional Graph



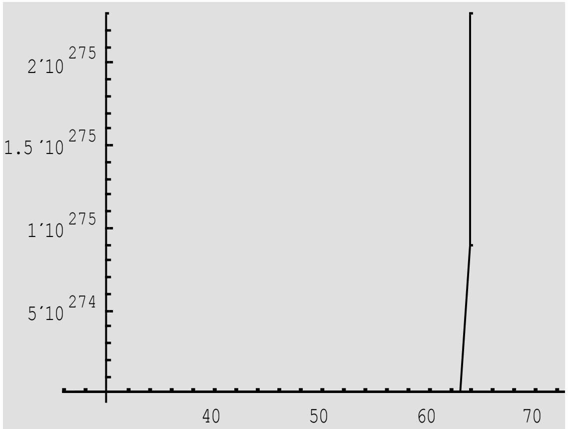
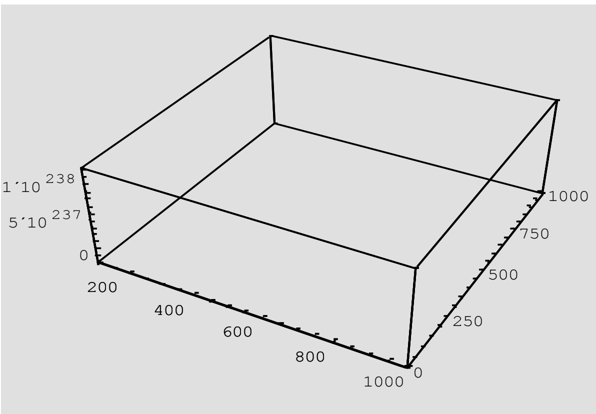


Figure 34 **Figure 34'**

Figure 34: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5, \beta = .6$, $\gamma = .7$ and $\lambda = 2$ and Values of the Variables $x \in (27,1000)$and $t \in (0,1000)$. Figure 34' Shows the Corresponding Two-Dimensional Graph

Eleventh Kind



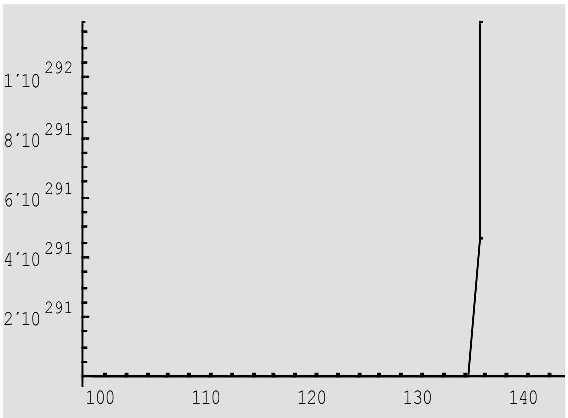
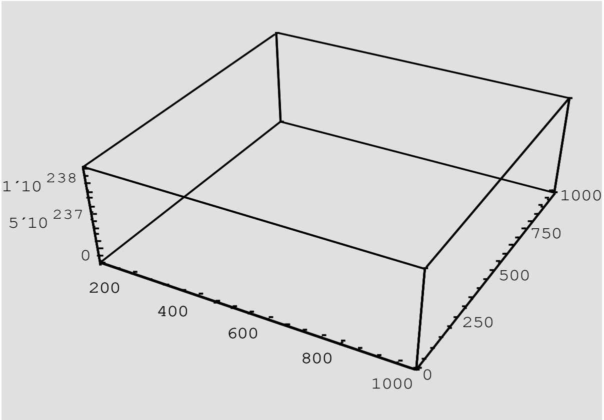


Figure 35 **Figure 35'**

Figure 35: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = 1, \beta = 1$, $\gamma = 1$and $\lambda = 1$ and Values of the Variables $x \in (99,1000)$and $t \in (0,1000)$. Figure 35' Shows the Corresponding Two-Dimensional Graph



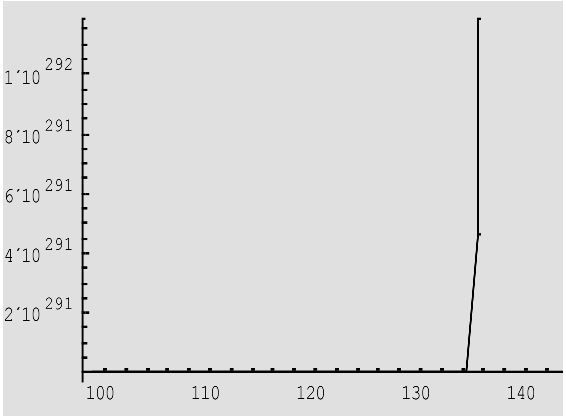
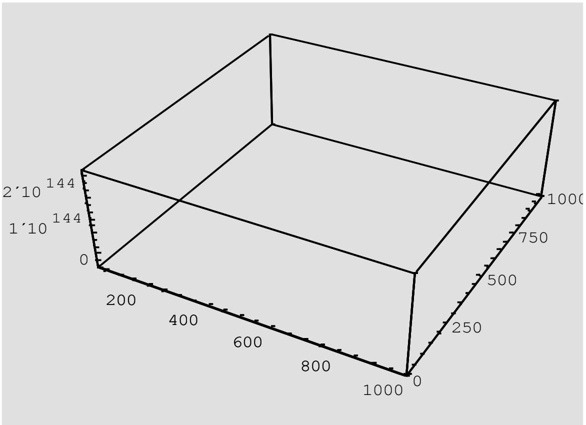


Figure 36 **Figure 36'**

Figure 36: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = -1, \beta = 1$ **,** $\gamma = 1$ **and** $\lambda = 1$ **and Values of the Variables** $x \in (99,1000)$ **and** $t \in (0,1000)$**. Figure 36' Shows the Corresponding Two-Dimensional Graph**



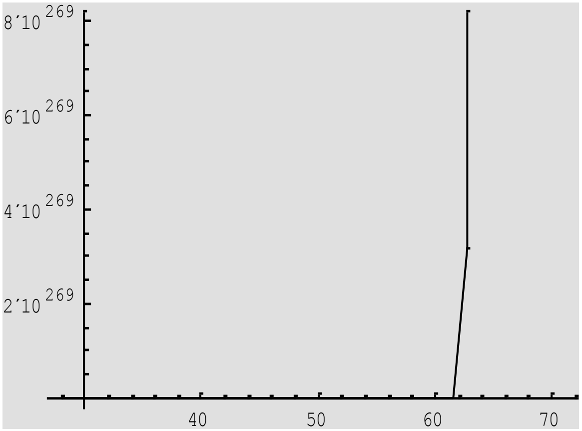


Figure 37 **Figure 37'**

Figure 37: The Surface Shows the Solution U (x, t) when the Values of the Parameters $\alpha = .5, \beta = .6$ **,** $\gamma = .7$ **and** $\lambda = 2$ **and Values of the Variables** $x \in (28,1000)$ **and** $t \in (0,1000)$**. Figure 37' Shows the Corresponding Two-Dimensional Graph**

RESULTS AND DISCUSSIONS

To find the nature of the graphs and to give the physical meaning of the solution of non-linear differential equation (10) we have drawn 16 numbers of three dimensional and two dimensional graphs for a wide or close range of values of the variables x and t and for different values of the parameters α, β, γ. These graphs have shown that the approximate solution (21) is valid for all values of x and t and for lower and higher parametric values of α, β and γ. These graphs also indicate that the shape of the graph depends on the values of the variables but not on parametric values. Here we can see four different shapes of the graph which are mostly depends on the range of the values of x and t but not on any parameter. We can classify these four kinds in the following way. Say for all parametric values but for close range of x and t that is if x is from 0 to 2.5 and t is from 0 to 0.2 then let us represent the graph by first kind, which are depicted in figures 1 to 5. Similarly, for a wide range values of x and t that means say x is from 0 to 1000 and t is also from 0 to 1000 the graphs are same which we call second kind which are depicted in figures 6 to 9. Another shape of the graph one can find for close range of x but wide range of t, which we call as third kind. That is the x values say is from 700 to 1000 but t is from 0 to 1000, which are depicted in figs. 10 to 12. But when the range of x is from 1 to 1.00001 and the range of t is from 0 to 1 or 0 to 1000 we have got another shape of the graphs and we call these types of the shape as fourth kind, which are depicted in figures of 13 to 16. The two-dimensional graphs that is for a fixed values of t but for a wide range of x the corresponding figures of first kind are depicted in figures from 1' to 5' and that of second kind are depicted in figures from 6' to 9' and the third kind are depicted in figures from 10' to 12' and the fourth kind are depicted in figures from 13' to 16'. All these graphs shows that the approximate solution that found by HPM method are most accurate and there is no chaotic solution in this type of highly non-linear differential equation with this type of initial conditions. Also it is clear from the graphs that the solution is valid for lower and higher values of the parameter, which is the main advantage of the HPM method.

The approximate solution of (22) that is for the same non-linear equation with the other type of initial conditions, the solution found by HPM method is given in equation (24). To find the nature of the solution we have drawn many three dimensional graphs for a wide or short range of values of the variables x and t and for different values of the parameters and corresponding two- dimensional graphs for fixed values of t. From these graphs we have chosen only 21 numbers of both three-dimensional and two-dimensional graphs to show the differences of shape of the graphs.

In the first case that is for the differential equation (10) we got only four types of different graphs, which are almost same but a minor difference in bend point, which is negligible. But in the second case that is for other type of initial conditions we got various types of graphs. This is mainly because of the different type of initial conditions. The main point here is that the solution of equation (22) exists for all values of the parameters except negative values of the parameter λ for a particular range of values of x and t which are depicted in figures from 17 to 24. After certain range of x and t values the solution does not exists. And these can be seen for the different range of the values of the x variables and the different values of the parameters. This can be easily understood if we look very carefully to the figures: 17 to 24, we can see that the solution exists only if x lies between 0 to 136 approximately but for all t and after that there is no solution. This one can see from the graphs in figure 35 to figure 37. For negative exponential initial conditions the solution exists in any range of values of x and t, which can be seen in figure 25. The solution (24) gives us some irregular shape of the graphs and also chaotic solution arises with these initial conditions, whereas in the solution (21) there was no such type of graphs found. This is only because of the initial conditions. The chaotic solution found for a very close range of x values but almost for all t and even for any parametric value. This one can find in the graphs of figures 26 to 29. Again if we compare figure 25, 30, 31 we can see that for $-\lambda$ that is for negative exponential the graph depends on x-values but not on t.

It is clear from these calculations that the solution is sensitive to the initial conditions. For the same equation due to the first type of initial conditions we got regular solution for all x and t. But in the second case we got both regular and irregular or chaotic solution. This proves the sensitivity of the initial conditions. From these solutions it is clear that HPM is a good semi-analytic approximation method to solve non-linear differential equation or dynamical system.

REFERENCES

1. He J. H, Some Asymptotic Methods for Strongly Nonlinear Equations, International Journal of Modern Physics B 20 (2006) 1141–1199.

2. He J.H, Non-Perturbative Methods for Strongly Nonlinear Problems, Dissertation.de-Verlag im Internet GmbH, Berlin, (2006).

3. Sajadi H., Ganj D.D. & Shenas Y.V., Application of Numerical semi-Analytical approach on Van der Pol-Duffing Oscillators, Journal of Advanced Research to Mathematical Engineering Vol-1, (2010/iss3), 136-141.

4. Njah A. N, Vincent U.E, Chaos Synchronization between Single and Double Wells Duffing-Van der Pol oscillators using active control, Chaos, Solitons & Fractals (2008); 37:1356-61.

5. Ji J., Zhang N., Additive Resonances of a Controlled Van der Pol-Duffing oscillator, J Sound Vibr (2008); 315:22-23.

6. Kimiaeifar, A., Saidi, A.R., Bagheri G.H., Rahimpour M., Domairry D.G., Analytical Solution for Van der Pol-Duffing Oscillators, Chaos, Solitons and Fractals 42 (2009) 2660-2666.

7. Ganji D.D, Sadighi, A., Application of He’s Homotopy- Perturbation Method to Nonlinear Coupled Systems of Relation-Diffusion Equations, Int. J. Nonlinear Sci. Numer, Simul, 7, (2006) 411-418.